International Journal of Theoretical Physics, Vol. 24, No. 12, 1985

Reproducing Kernel Hilbert Spaces and Extremal Problems for Scattering of Particles with Arbitrary Spins

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Received June 16, 1985

In this paper each helicity amplitude of the two-body scattering of particles with arbitrary spins is considered as an element of a special class of Hilbert spaces $H^{[\mu]}$. This space, which is called reproducing kernel Hilbert space (RKHS) has many special properties that appear to make it a natural space of functions to associate with the scattering helicity amplitudes. Some of the special properties of the RKHS are developed and then used to characterization of reproducing kernel (RK) of $H^{[\mu]}$ as the solution to certain extremal problems. Then, it was shown that the optimal scattering state from the RKHS of the helicity amplitudes is analogous to the coherent state from the RKHS of the wave functions. The essential characteristic features of the scattering of particles with arbitrary spins in the optimal state dominance limit are established. An important alternative to the partial wave helicity analysis in terms of a fundamental set of optimal states is presented.

1. INTRODUCTION

The *reproducing kernel Hilbert spaces* (RKHS) were first studied by Moore (1916, 1935, 1939) in connection with a general theory of integral equation. M. G. Krein (1940, 1949, 1963) used them in his fundamental studies on the extension of positive-definite functions (see also Devinatz, 1953, 1954). In fact a systematic abstract theory of RKHS has been developed by Aronszajn (1943, 1950). They were also encountered and used effectively in the theory of boundary value and related problems (Bergman and Schiffer, 1953), conformal mapping (Bergman, 1950; Nehari, 1952), numerical analysis (Davis, 1963; Golomb and Weinberger, 1958; Larkin, 1970; Richter, 1971; Mansfield, 1971), group representations (see, e.g., Carey, 1977, 1978), coherent state (Bargmann, 1961; Klauder and McKenna, 1964; Klauder and Sudershan, 1968; Perelomov, 1972) and elementary particle physics

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0020-7748/85/1200-1217$04.50/0

(Cutkosky, 1973; Okubo, 1974; Ion and Scutaru, 1985). Perelomov was the first to note that the notion of coherent state (Glauber, 1963a, 1963b; Klauder and McKenna, 1964) and the reproducing kernel of the Hilbert space of wave function are the same. There are, of course, several sources that are basic to subject of RKHS from which the reader may gain additional insight into RKHS methods. Notable among these are Meschkowski (1962), Parzen (1967), Shapiro (1971), and Hille (1972).

The present paper may be considered as a continuation and an extension of our previous paper (Ion and Scutaru, 1985) in which the two-body scattering amplitudes is considered as an element of a RKHS. Then we have shown that the RKHSs are adequate variational spaces for the description of the scattering in terms of an optimum principle (Ion, 1982a,b). We have pointed out that the notion of the optimal scattering states and the reproducing kernel of the Hilbert space of the scattering amplitudes are the same. Then, we have indicated the unified manner in which the class of the dual diffractive scattering (DDS) (Ion, 1981a,b) as well as the dual diffractive resonances (DDR) phenomena (Ion, 1981b; Ion and Ion-Mihai, 1981a,b) are described by the reproducing kernels.

Our investigations are concerned with the optimal state description of scattering of particles with arbitrary spins. So, in Section 2 we shall discuss briefly the essential definitions and results on RKHS and we give a further development and extension of the RKHS properties to the Hilbert spaces of helicity amplitudes. In Section 3 the general solutions of some extremal problems for scattering of particles with arbitrary spins are given. Concrete applications when the scattering helicity amplitudes are elements of finite-dimensional subspace of $L^2(-1, +1)$ are also presented in Section 3, while the conclusions are summarized in Section 4.

2. THE RKHS ASSOCIATED WITH HELICITY AMPLITUDES

Let us consider the two-body elastic scattering

$$a+b\rightarrow a+b \tag{1}$$

where the particles a and b have the spins S_a and S_b, respectively. For the description of the system (1) we shall use the helicity formalism of Jacob and Wick (1959). Therefore, let $f^{[\mu]}(x)\equiv\langle\mu'_a\mu'_b|F(s, t)|\mu_a\mu_b\rangle$, be the helicity amplitudes of the system (1) with the initial helicities μ_a, μ_b and final helicities μ'_a, μ'_b, where s and t are the squares of c.m. energy and momentum transfer variables while $x\equiv\cos\theta$, with θ the c.m. angle. The normalization is chosen in such a way that the differential cross section $d\sigma/d\Omega^{[\mu]}(x)$ for a given channel $[\mu] \equiv (\mu_a\mu_b;\ \mu'_a\mu'_b)$ is given by

$$\frac{d\sigma^{[\mu]}}{d\Omega}(x) = |f^{[\mu]}(x)|^2, \qquad x \in [-1, +1] \tag{2}$$

$$\sigma_{\rm el}^{[\mu]} = 2\pi \int_{-1}^{+1} |f^{[\mu]}(x)|^2\, dx = 2\pi \|f^{[\mu]}\|^2 \tag{3}$$

Since we will work at fixed energy, the dependence of $f^{[\mu]}(x)$, $d\sigma/d\Omega^{[\mu]}(x)$ and $\sigma_{\rm el}^{[\mu]}$ on this variable was suppressed.

Let $[d\sigma/d\Omega](x)$, $\sigma_{\rm el}$ and $\sigma_{\rm T}$ be the unpolarized (differential, elastic integrated and total) cross sections, respectively. Then we have

$$\frac{d\sigma}{d\Omega}(x) = \frac{1}{(2S_a+1)(2S_b+1)} \sum_{[\mu]} \frac{d\sigma^{[\mu]}}{d\Omega}(x), \qquad x \in [-1, +1] \tag{4}$$

$$\sigma_{\rm el} = \frac{1}{(2S_a+1)(2S_b+1)} \sum_{[\mu]} \sigma_{\rm el}^{[\mu]} \tag{5}$$

$$\sigma_{\rm T} = \frac{1}{(2S_a+1)(2S_b+1)} \sum_{[\mu_0]} \sigma_{\rm T}^{[\mu_0]} \tag{6}$$

where $[\mu_0] \equiv (\mu_a\mu_b;\ \mu_a\mu_b)$. For the $[\mu_0]$ elastic channels $\operatorname{Im} f^{[\mu_0]}(1)$ and $\sigma_{\rm T}^{[\mu_0]}$ are related via optical theorem

$$\sigma_{\rm T}^{[\mu_0]} = \frac{4\pi}{k} \operatorname{Im} f^{[\mu_0]}(1) \tag{7}$$

k being the c.m. momentum.

Now, an important step in description of the scattering in terms of an optimum principle is to consider that each helicity amplitude $f^{[\mu]}$ is an element of a functional Hilbert space defined on the interval $S \equiv [-1, +1]$ with the inner product $\langle\ ,\ \rangle$ and norm $\|\ \|$ given by

$$\langle f^{[\mu]}, g^{[\mu]} \rangle = \int_{-1}^{+1} f^{[\mu]}(x)\overline{g^{[\mu]}}(x)\, dx \tag{8a}$$

$$\|f^{[\mu]}\|^2 = \langle f^{[\mu]}, f^{[\mu]} \rangle < \infty \tag{8b}$$

Definition 1. A functional Hilbert space $H^{[\mu]}$ is a Hilbert space of complex-valued functions on a (nonempty) set S with two natural requirements: (i) the evaluation functionals on $H^{[\mu]}$ are linear, and (ii) the evaluation functionals on $H^{[\mu]}$ are bounded, i.e., to each y in S there correspond a positive finite constant $C_y^{[\mu]}$, such that

$$|f^{[\mu]}(y)| \leq C_y^{[\mu]} \|f^{[\mu]}\|, \qquad \text{all}\ f^{[\mu]} \in H^{[\mu]} \tag{9}$$

Now, we shall recall briefly some of the essential features of the reproducing kernel Hilbert spaces (RKHS) which we shall use in this investigation.

Definition 2. A Hilbert space H of complex-valued functions defined on a set S is called RKHS if it enjoys the following reproducing property. There exists a complex-valued function $K(x, y)$ on $S \times S$, called reproducing kernel (RK), such that (i) for any fixed $y \in S$, K_y is in H, (ii) $K_y(x) = K(x, y)$ induces the reproducing property

$$\langle f, K_y \rangle = f(y) \tag{10}$$

for each $f \in H$, and any $y \in S$. K_y is called the *reproducing element* for the point y, while the totality of reproducing elements is RK of H.

Theorem 1. A Hilbert space is a RKHS if and only if it is a functional Hilbert space.

This result, which is a consequence of the Frechet-Riesz representation theorem [see Davis (1963), Theorem 9.3.3], tells us that a Hilbert function space H is an RKHS if and only if the evaluation functional is bounded [see Higgins (1977)].

Corollary 1. If each $f^{[\mu]} \in H^{[\mu]}$ is continuous, then $H^{[\mu]}$ is a RKHS. The converse of this corollary is not true. Letho (1952) has given an example of RKHS which contains discontinuous functions.

Most of the important properties of RKHS are discussed by Aronszajn (1950), Parzen (1958, 1967), Hajek (1962), and Meschkowski (1962). In the following a few of the RKHS properties are listed. In order to give some idea of the simplifications made possible by the reproducing property (10) some of the elementary proofs are given here.

(A) *Autoreproducing Properties.* Let K be the RK of H defined on S. Then

$$\langle K_y, K_x \rangle = K_y(x) = K(x, y), \qquad \|K_y\|^2 = K(y, y) \tag{11a}$$

$$K(x, y) = \overline{K(y, x)}, \qquad |K(x, y)|^2 \le K(x, x)K(y, y) \tag{11b}$$

for any x, y in S.

Proof. These properties follow readily from the definition (10b).

(B) *Uniqueness of RK.* If a RKHS has two RK K and R, they must be identical.

Proof:

$$\begin{aligned}\|K_y - R_y\|^2 &= \langle K_y - R_y, K_y \rangle - \langle K_y - R_y, R_y \rangle \\ &= K(y, y) - R(y, y) - K(y, y) + R(y, y) = 0 \quad \blacksquare\end{aligned}$$

(C) *Uniqueness of RKHS.* Two RKHS H and G both defined on S, which have the same RK K, must be identical.

Proof. Let $\langle\ ,\ \rangle_H$ and $\langle\ ,\ \rangle_G$ be the corresponding inner products in H and G, respectively. Then, for every $(x, y) \in S \times S$ we write

$$\langle K_y, K_x\rangle_H = K(x, y) = \langle K_y, K_x\rangle_G$$

From this one can show [see Parzen (1958), congruence theorem] that there exists an isometric mapping ψ from H to G with the property $\psi(K_y) = K_y$. Now, if g in G correspond under ψ to f in H, then $f(y) = g(y)$ for every y in S, because

$$f(y) = \langle f, K_y\rangle_H = \langle g, K_y\rangle_G = g(y) \quad \blacksquare$$

(D) *Completeness of* $\{K_y, y \in S\}$. Let K be the RK of H on S, then the set $\{K_y, y \in S\}$ is complete in H, i.e., their finite linear combinations are dense in H.

Proof. The reproducing property $\langle g, K_y\rangle = g(y)$, $y \in S$, implies that only vector g orthogonal to each K_y, $y \in S$, is $g = 0$. $\blacksquare$

(E) *Pointwise, Weak and Strong Convergence.* Let K be the RK of the RKHS H defined on S. Then, the *weak* and, hence, *strong convergence* of a sequence $\{f_n\}$ to f in H implies the *pointwise* convergence of $f_n(y)$ to $f(y)$ for any y in S. The convergence is uniform over any subset of S for which $K(y, y)$ is bounded.

Proof. By Schwartz inequality and the reproducing property (10) we have

$$\begin{aligned}|f_n(y) - f(y)| &= |\langle f_n - f, K_y\rangle| \le \|f_n - f\|\,\|K_y\| \\ &= \|f_n - f\|[K(y, y)]^{1/2} \to 0 \qquad \text{as } n \to \infty. \quad \blacksquare\end{aligned}$$

(F) *Positiveness of* $K(x, y)$. If K is the RK of H defined on S, then $K(x, y)$ is a function positive-definite on $S \times S$, i.e.,

$$\sum_{i,j}^{N} \overline{a_i} a_j K(y_i, y_j) \ge 0 \tag{12}$$

for all finite sets $\{a_i\} \subset C$ and $\{y_i\} \subset S$.

Proof. We give the integer N, $\{a_i\} \subset C$ and $\{y_i\} \subset S$, $i = 1, \ldots, N$, and we define the vector $\phi = \sum_{i=1}^{N} a_i K_{yi}$ in H [see property (D)]. Then, we write

$$0 \le \langle \phi, \phi\rangle = \sum_{i,j} \bar{a}_i a_j \langle K_{yj}, K_{yi}\rangle = \sum_{i,j}^{N} \bar{a}_i a_j K(y_i, y_j) \ge 0$$

since $\langle K_{yj}, K_{yi}\rangle = K(y_i, y_j)$. $\blacksquare$

(*G*) *Minimum Norm Property of RK.* If K is the RK of H defined on S, then the function $f = bK_y$, $b = a[K(y,y)]^{-1}$, $K(y,y) \neq 0$, is the unique solution of the minimum norm problem: Minimize $\|f\|$ subject to $f \in H$, $f(y) = a$, $a \in C$, $y \in S$.

Proof. Using (10) and Schwarz inequality we get

$$|f(y)| = |\langle f, K_y\rangle| \leq \|K_y\| \|f\| = [K(y,y)]^{1/2}\|f\| \tag{13}$$

for all $f \in H$ and $y \in S$, the equality holding in (12) if and only if

$$f(x) = f(y)\frac{K(x,y)}{K(y,y)} = a\frac{K(x,y)}{K(y,y)} = bK_y(x). \quad \blacksquare \tag{14}$$

(*H*) *Smoothness.* Let K be the RK of H defined on S. If K is a continuous function on $S \times S$, then the functions in H are continuous on S.

Proof. We write

$$\begin{aligned} |f(x) - f(y)| &= |\langle f, K_x - K_y\rangle| \leq \|f\| \|K_x - K_y\| \\ &= \|f\|\{K(x,x) - 2K(x,y) + K(y,y)\} \to 0 \qquad \text{as } x \to y \end{aligned}$$

for any x, y, in S, since K is continuous. $\blacksquare$

(*I*) *Calculation of RK.* Let K be the RK of H defined on S. If $\{\phi_n\}$ is a complete orthonormal sequence in H, then

$$K(x,y) = \sum_n \phi_n(x)\bar{\phi}_n(y) \tag{15}$$

Proof. We write $K_y(x) = K(x,y)$. Then, for each $y \in S$, we consider the expansion

$$K_y = \sum_n \langle K_y, \phi_n\rangle \phi_n = \sum_n \overline{\phi_n(y)}\phi_n$$

since $\langle K_y, \phi_n\rangle = \overline{\phi_n(y)}$. $\blacksquare$

(*J*) *Overcompleteness of* $\{K_y, y \in S\}$. Let K be the RK of H defined on S. Then, the full set $\{K_y, y \in S\}$ is overcomplete in H. That is, there must be certain linear dependencies among the vectors K_y, i.e., only a subset of $\{K_y, y \in S\}$ span H.

Proof. This property is connected with the "resolution of unity" of which real meaning is embodied in the following integral representation

$$\langle f, g\rangle = \int_{-1}^{+1} \langle f, K_t\rangle\langle K_t, g\rangle\, dt, \qquad \text{all } f, g \in H \tag{16}$$

from which we get

$$K_y = \int_{-1}^{+1} \langle K_y, K_t \rangle K_t \, dt \tag{17}$$

which express K_y in terms of all the reproducing elements K_t, $t \in S$. This is one of the most characteristic properties of the overcomplete set $\{K_y, y \in S\}$. The validity of (16) is obtained using the definition (8a) and the reproducing property (10). ■

Remark 1. The denseness of $\{K_y, y \in S\}$ shows that the RKHS H can be regarded as a closure of the linear space G of functions of form $\sum_i \alpha_i K_{yi}$, $\{\alpha_i\} \subset C$ and $\{y_i\} \subset S$, while the property (E) shows that this closure can be obtained by taking pointwise limits of sequences in G rather than limits in the RK norm.

3. OPTIMAL STATES VERSUS REPRODUCING KERNELS

Now, the helicity amplitudes $f^{[\mu]}$ will be determined by optimization methods assuming that the system (1) behaves so as to minimize the integrated unpolarized elastic cross section subject to some constraints. This approach, by which the system is completely specified by identifying the criterion of effectiveness and applying optimization to it, is known as describing the system in terms of an optimum principle.

Therefore, let us consider that each helicity amplitude $f^{[\mu]}$ is an element of a RKHS $H^{[\mu]}$ defined on the interval $[-1, +1]$.

Corollary 2. If $K^{[\mu]}$ is the RK of $H^{[\mu]}$, $f^{[\mu]} \in H^{[\mu]}$, then for each $y \in [-1, +1]$ for which $f^{[\mu]}(y) \neq 0$, $K^{[\mu]}(y, y) \neq 0$, the functionals (2) and (3) must obey the inequality

$$\frac{d\sigma^{[\mu]}}{d\Omega}(y) \leq \frac{\sigma_{\text{el}}^{[\mu]}}{2\pi} K^{[\mu]}(y, y) \tag{18}$$

the equality holding in (18) if and only if

$$f^{[\mu]}(x) = f^{[\mu]}(y) \frac{K^{[\mu]}(x, y)}{K^{[\mu]}(y, y)}, \qquad x \in [-1, +1] \tag{19}$$

Definition 3. The scattering state of the system (2.1) described by the helicity amplitude (19) is called *optimal state* of the channel $[\mu]$.

Remark 2. If in (19) we put $f^{[\mu]}(y) = K^{[\mu]}(y, y)$, then $f^{[\mu]}(x) = K^{[\mu]}(x, y)$. Hence the notion of *optimal state* and the reproducing kernel of $H^{[\mu]}$ are the same. With this respect the optimal state from the RKHS of the scattering amplitudes is analogous to the *coherent state* from the RKHS of the wave function.

The following corollary lists different equivalent extremal properties of the optimal states (19).

Corollary 3. If $K^{[\mu]}$ is the RK of $H^{[\mu]}$, and if $K^{[\mu]}(y, y) \neq 0$, $y \in [-1, 1]$, then (i) functions of form $b^{[\mu]}K_y^{[\mu]}$ minimize $\|f^{[\mu]}\| \, |f^{[\mu]}(y)|^{-1}$ subject to $f^{[\mu]} \in H^{[\mu]}$, $f^{[\mu]}(y) \neq 0$, and maximize $|f^{[\mu]}(y)| \, \|f^{[\mu]}\|^{-1}$ subject to $f^{[\mu]} \in H^{[\mu]}$, $f^{[\mu]} \neq 0$, (ii) functions of form $b^{[\mu]}K_y^{[\mu]}$, where $|b|^2 = [K^{[\mu]}(y, y)]^{-1}$, maximize $|f^{[\mu]}(y)|$ subject to $f^{[\mu]} \in H^{[\mu]}$, $\|f^{[\mu]}\| = 1$.

Next, let us define the following extremal problem.

Problem A. Minimize σ_{el} subject to $f^{[\mu]} \in H^{[\mu]}$, when $[d\sigma/d\Omega](y)$, $y \in [-1, +1]$, is fixed. In order to solve problem A we use the Lagrange multiplier method [Wilde and Beightler (1967)] which in essence is based on transformation of the constrained minimization problem into an unconstrained problem. Therefore, let $\{e_n^{[\mu]}\}$ be a complete orthonormal sequence in the RKHS $H^{[\mu]}$ with the RK given by

$$K^{[\mu]}(x, y) = \sum_n e_n^{[\mu]}(x) \, \overline{e_n^{[\mu]}}(y) \tag{20}$$

and consider the expansion

$$f^{[\mu]}(x) = \sum_n C_n^{[\mu]} e_n^{[\mu]}(x), \qquad C_n^{[\mu]} = \langle f^{[\mu]}, e_n \rangle, \qquad C_n \in C \tag{21}$$

Then, the functionals (4), (5) are given by

$$\frac{d\sigma}{d\Omega}(x) = \frac{1}{(2S_a+1)(2S_b+1)} \sum_{[\mu]} \left| \sum_n C_n^{[\mu]} e_n^{[\mu]}(x) \right|^2 \tag{22a}$$

$$\sigma_{\text{el}} = \frac{1}{(2S_a+1)(2S_b+1)} \sum_{[\mu]} \sum_n |C_n^{[\mu]}|^2 \tag{22b}$$

Now, we introduce the variational function

$$\mathscr{L} = \sum_{[\mu]} \sum_n |C_n^{[\mu]}|^2 + \alpha \left\{ (2S_a+1)(2S_b+1) \frac{d\sigma}{d\Omega}(y) - \sum_{[\mu]} \left| \sum_n C_n^{[\mu]} e_n^{[\mu]}(y) \right|^2 \right\}$$

where α is a real Lagrange multiplier, and problem A is reduced to the unconstrained minimization problem

$$\mathscr{L}(C_1^{[\mu]}, C_2^{[\mu]}, \ldots, C_n^{[\mu]}, \ldots, \alpha) \to \text{minimum} \tag{23}$$

Now, if the stationary solution of the problem (23) is denoted by $\overset{*}{C}{}_n^{[\mu]}$, $\overset{*}{\alpha}$, then from the variational equations we obtain

$$\overset{*}{C}{}_n^{[\mu]}=\overset{*}{\alpha}\overset{*}{f}{}^{[\mu]}(y) \tag{24a}$$

$$\overset{*}{\alpha}=\frac{1}{K^{[\mu]}(y,y)}=\frac{\overset{*}{\sigma}_{\text{el}}}{2\pi(d\sigma/d\Omega)(y)} \tag{24b}$$

$$\overset{*}{f}{}^{[\mu]}(x)=\overset{*}{f}{}^{[\mu]}(y)\frac{K^{[\mu]}(x,y)}{K^{[\mu]}(y,y)} \tag{24c}$$

for all $[\mu]$ and $y\in[-1,+1]$ for which $f^{[\mu]}(y)\neq 0$ and $K^{[\mu]}(y,y)\neq 0$.

On the other hand one can verify that the Lagrange multiplier $\overset{*}{\alpha}$ satisfies the condition that the Hessian corresponding to second derivatives of $\mathscr{L}$ is a nonnegative definite matrix for minimum. Then we have obtained the following result.

Theorem 2. If K is the RK of the RKHS $H^{[\mu]}$ and $f^{[\mu]}\in H^{[\mu]}$ then the functionals (4) and (5) must obey the bounds

$$\frac{d\sigma}{d\Omega}(y)\leq\frac{\sigma_{\text{el}}}{2\pi}K^{[\mu]}(y,y) \tag{25}$$

for all $[\mu]$ and $y\in[-1,+1]$ for which $f^{[\mu]}(y)\neq 0$ and $K^{[\mu]}(y,y)\neq 0$. The equality holds in (25) if and only if

$$f^{[\mu]}(x)=f^{[\mu]}(y)\frac{K^{[\mu]}(x,y)}{K^{[\mu]}(y,y)}=f^{[\mu]}(y)\frac{\sigma_{\text{el}}}{2\pi(d/d\Omega)(y)}K^{[\mu]}(x,y) \tag{26}$$

In particular, when the set $\{e_n\}$ in $H^{[\mu]}$ is the set of usual rotation functions $\{d^j_{\mu\nu}(x), x\in[-1,+1]\}$ [see Rose (1957), Edmonds (1957)], then the helicity scattering amplitudes $f^{[\mu]}$ are written in terms of partial amplitudes $f_j^{[\mu]}$ as

$$f^{[\mu]}(x)=\sum_{J_{\min}}^{J_\mu}(2j+1)f_j^{[\mu]}d^j_{\mu\nu}(x) \tag{27a}$$

where

$$J_{\min}\equiv\max\{|\mu|,|\nu|\},\qquad \mu=\mu_a-\mu_b,\ \nu=\mu'_a-\mu'_b;,\qquad [\mu]\equiv(\mu_a\mu_b;\ \mu'_a\mu'_b) \tag{27b}$$

Then, one can verify that (i) the helicity amplitudes $f^{[\mu]}$ is an element of a RKHS $H^{[\mu]}$ defined on $[-1,+1]$ if and only if $J_\mu<\infty$, and (ii) $H^{[\mu]}$

possess the reproducing kernel

$$K^{[\mu]}(x, y) = \sum_{J_{\min}}^{J_\mu} (j+\tfrac{1}{2}) d^j_{\mu\nu}(x) d^j_{\mu\nu}(y)$$

$$= \frac{[(J_\mu+1)^2-\mu^2]^{1/2}[(J_\mu+1)^2-\nu^2]^{1/2}}{2(J_\mu+1)} \times \frac{d^{J_\mu+1}_{\mu\nu}(x) d^{J_\mu}_{\mu\nu}(y) - d^{J_\mu}_{\mu\nu}(x) d^{J_\mu+1}_{\mu\nu}(y)}{x-y} \tag{28a}$$

$$K^{[\mu]}(y, y) = \frac{[(J_\mu+1)^2-\mu^2]^{1/2}[(J_\mu+1)^2-\nu^2]^{1/2}}{2(J_\mu+1)} \times \{\dot{d}^{J_\mu+1}_{\mu\nu}(y) d^{J_\mu}_{\mu\nu}(y) - \dot{d}^{J_\mu}_{\mu\nu}(y) d^{J_\mu+1}_{\mu\nu}(y)\} \tag{28b}$$

where $\dot{d}^J_{\mu\nu}(x) = d\, d^J_{\mu\nu}(x)/dx$.

Theorem 3. Assume that each helicity amplitude $f^{[\mu]}$ is an element of a Hilbert space $H^{[\mu]}$ which possesses the reproducing kernel (28a,b).

(i) Then, if σ_{el} and $(d\sigma/d\Omega)$ (1) are given, any cutoff J_μ on the total angular momentum must obey the bound

$$(J_\mu+1)^2 \geq \frac{4\pi}{\sigma_{\text{el}}} \cdot \frac{d\sigma}{d\Omega}(1) + \mu^2 \tag{29}$$

(ii) The equality holds in (29) if and only if $f^{[\mu]}(x)$ is the optimal amplitude

$$f^{[\mu]}(x) = f^{[\mu]}(1) \frac{K^{[\mu]}(x, 1)}{K^{[\mu]}(1, 1)} = f^{[\mu]}(1) \frac{1}{J_\mu+1} \frac{d^{J_\mu+1}_{\mu\mu}(x) - d^{J_\mu}_{\mu\mu}(x)}{x-1} \tag{30a}$$

where

$$J_\mu = \text{integer (or half-integer)}\left\{\left[\frac{4\pi}{\sigma_{\text{el}}}\frac{d\sigma}{d\Omega}(1)+\mu^2\right]^{1/2} - 1\right\} \tag{30b}$$

(iii) The logarithmic slope of the forward diffractive peak is indpeendent of $[\mu]$ and is given by

$$b_0 = \frac{d}{dt}\left[\ln \frac{d\sigma}{d\Omega}(s, t)\right]\bigg|_{t=0} = \frac{\lambda\!\!\!^{-2}}{4}\left[\frac{4\pi}{\sigma_{\text{el}}}\frac{d\sigma}{d\Omega}(1) - 1\right] = b_0^{[\mu]} \tag{31}$$

(iv) The forward diffraction peak of the optimal phenomena described by equations (30a,b) possesses the scaling property

$$\frac{1}{(d\sigma/d\Omega)(1)}\frac{d\sigma}{d\Omega}(x)=\left[\frac{J_1(\tau)}{\tau}\right]^2 \tag{32a}$$

with the scaling variable

$$\tau=2[|t|b_0]^{1/2}=\left\{\lambda^2|t|\left[\frac{4\pi}{\sigma_{\text{el}}}\frac{d\sigma}{d\Omega}(1)-1\right]\right\}^{1/2} \tag{32b}$$

where $J_1(\tau)$ is the Bessel function of first order.

Proof. The results (i) and (ii) are obtained by introducing the kernel (28a,b) in (25) and (26). An important step here is the equality [see equation (24b)]

$$\begin{aligned}\frac{4\pi}{\sigma_{\text{el}}}\frac{d\sigma}{d\Omega}(1)&=2K^{[\mu]}(1,1)=(J_\mu+1)^2-\mu^2\\&=\frac{4\pi}{\sigma_{\text{el}}^{[\mu]}}\frac{d\sigma^{[\mu]}}{d\Omega}(1)\end{aligned} \tag{33}$$

for all $[\mu]$ for which $K^{[\mu]}(1,1)\neq 0$. Then, the results (iii) and (iv) are derived using the definitions (4), (31), the equality (33) and the following relations:

$$d^J_{\mu\nu}(x)=J_{|\mu-\nu|}\left[(2J+1)\sin\frac{\theta}{2}\right] \tag{34}$$

where $J_{|\mu-\nu|}$ are the Bessel functions of $|\mu-\nu|$ order. The equality $b_0^{[\mu]}=b_0$ for each $[\mu]$ with $\mu=\nu$ are direct consequences of the relation (33). Also, we have used the results

$$\frac{d}{dx}\left[\frac{K^{[\mu]}(x,1)}{K(1,1)}\right]\Bigg|_{x=1}=\frac{1}{2(J_\mu+1)}[\ddot{d}^{J_\mu+1}_{\mu\mu}(1)-\ddot{d}^{J_\mu}_{\mu\mu}(1)]=\frac{J_\mu(J_\mu+2)-\mu^2}{4} \tag{35a}$$

since

$$\begin{aligned}\ddot{d}^J_{\mu\mu}(1)&=\frac{d^2}{dx}[d^J_{\mu\mu}(x)]\Bigg|_{x=1}\\&=\frac{\mu(\mu-1)}{2}+\frac{\mu}{2}[J(J+1)-\mu(\mu+1)]\\&\quad+\tfrac{1}{8}[J(J+1)-(\mu+1)(\mu+2)][J(J+1)-\mu(\mu+1)]\end{aligned} \tag{35b}$$

We note that (31) is an exact result while (32a,b) is valid only for large values J_μ [given by (30b)] and small angles. ■

Theorem 4. Assume that each helicity amplitude $f^{[\mu]}$ is an element of RKHS $H^{[\mu]}$ with RK (28a,b) and that the points $y_i \in [-1, +1]$, $i = 1, \ldots, N$ are the zeros of the Jacobi polynomial corresponding to the rotation function $d_{\mu\nu}^{J_\mu+1}(x)$.

(i) Then, the set

$$\frac{K^{[\mu]}(x, y_i)}{K^{[\mu]}(y_i, y_i)} = \frac{d_{\mu\nu}^{J_\mu+1}(x)}{(x-y_i)\dot{d}_{\mu\nu}^{J_\mu+1}(y_i)}, \qquad i = 1, \ldots, N \tag{36}$$

called fundamental optimal states (FOS) system, is a complete orthogonal set.

(ii) The helicity amplitude $f^{[\mu]}(x)$ is expanded in terms of these optimal states as follows:

$$\begin{aligned} f^{[\mu]}(x) &= \sum_{i=1}^{N} f(y_i) \frac{K^{[\mu]}(x, y_i)}{K^{[\mu]}(y_i, y_i)} \\ &= \sum_{i=1}^{N} f(y_i) \frac{d_{\mu\nu}^{J_\mu+1}(x)}{(x-y_i)\dot{d}_{\mu\nu}^{J_\mu+1}(y_i)} \end{aligned} \tag{37}$$

and

$$\begin{aligned} \frac{\sigma_{\text{el}}^{[\mu]}}{2\pi} &= \sum_{i=1}^{N} \frac{d\sigma^{[\mu]}}{d\Omega}(y_i) \frac{1}{K^{[\mu]}(y_i, y_i)} \\ &= \frac{2(J_\mu+1)}{[(J_\mu+1)^2-\mu^2]^{1/2}[(J_\mu+1)^2-\nu^2]^{1/2}} \\ &\quad \times \sum_{i=1}^{N} \frac{(d\sigma/d\Omega)^{[\mu]}(y_i)}{\dot{d}_{\mu\nu}^{J_\mu+1}(y_i) d_{\mu\nu}^{J_\mu}(y_i)} \end{aligned} \tag{38}$$

(iii) The partial helicity amplitudes are expressed in terms of $f(y_i)$, $i = 1, \ldots, N$, as follows:

$$\begin{aligned} f_j^{[\mu]} &= \frac{(J_\mu+1)}{[(J_\mu+1)^2-\mu^2]^{1/2}[(J_\mu+1)^2-\nu^2]^{1/2}} \\ &\quad \times \sum_{i=1}^{N} f^{[\mu]}(y_i) \frac{d_{\mu\nu}^{j}(y_i)}{\dot{d}_{\mu\nu}^{J_\mu+1}(y_i) d_{\mu\nu}^{J_\mu}(y_i)} \end{aligned} \tag{39}$$

for all $[\mu]$ and $j \le J_\mu$, and $f_j^{[\mu]} = 0$ for $j > J_\mu$.

(iv) The helicity amplitude (37) is the element of $H^{[\mu]}$ of least norm where $f^{[\mu]}(y_i)$ are given for all $y_i \in [-1, +1]$, $i = 1, 2, \ldots, N$:

Proof. The results (i)-(iii) are obtained from the property (D) [see Section 2], equations (28a,b) and

$$\langle K_{y_j}^{[\mu]}, K_{y_i}^{[\mu]}\rangle = K_{y_j}^{[\mu]}(y_i) = K^{[\mu]}(y_i, y_j) = K^{[\mu]}(y_i, y_i)\delta_{ij} \tag{40}$$

The result (iv) is proved by solving the following optimization problem: find an element of $H^{[\mu]}$ of least norm when $f^{[\mu]}(y_i)$, $i=1, \ldots, N$, are given. The general solution of this problem is [see Ion and Scutaru (1985), Theorem 4]:

$$f^{[\mu]}(x) = \sum_{i=1}^{N} f(y_i)\frac{\Delta_{Ni}^{[\mu]}}{\Delta_N^{[\mu]}}, \qquad \Delta_N^{[\mu]} \neq 0 \tag{41}$$

where $\Delta_N^{[\mu]} = \det[K^{[\mu]}(y_i, y_j)]$, $i, j = 1, 2, \ldots, N$, while $\Delta_{Ni}^{[\mu]}$ is the determinant obtained from $\Delta_N^{[\mu]}$ by the substitution of the row i with the following elements: $K^{[\mu]}(x, y_1)$, $K^{[\mu]}(x, y_2), \ldots, K^{[\mu]}(x, y_N)$. Then, since the points $y_i \in [-1, +1]$ in our case satisfy the conditions (40) we get the results (iv). ■

Remark 3. For the RKHS $H^{[\mu]}$ with the RK (28a,b) the FOS system (36) is the system of fundamental polynomials [see Davis, 1963, p. 33] for the Lagrange interpolation when the helicity amplitude is given in the zeros of Jacobi polynomials corresponding to the rotation functions $d_{\mu\nu}^{J_\mu+1}(x)$.

4. CONCLUSIONS

In this paper we have applied the RKHS method to the two-body scattering of particles with arbitrary spins. Then, as a first step in description of the scattering in terms of an optimum principle, we have considered that each helicity amplitude $f^{[\mu]}$ is an element of a functional Hilbert space $H^{[\mu]}$ defined on $[-1, +1]$. Then, since the linear functionals are bounded for each $y \in [-1, +1]$, the Hilbert space $H^{[\mu]}$ is a RKHS (see Definition 2, and Theorem 1).

The conclusions of this paper may be summarized as follows.

(1°) The RKHS's are a special class of Hilbert spaces that have many special properties such as (a) autoreproducing of RK, (b) uniqueness of RK, (c) uniqueness of RKHS, (d) completeness of the set $\{K_y, y \in [-1, +1]\}$, (e) pointwise convergence in RKHS's, (f) positiveness of RK, (g) minimum norm of RK, (h) smoothness, (j) overcompleteness of the full set $\{K_y, y \in [-1, +1]\}$.

(2°) The RK of the Hilbert space $H^{[\mu]}$ of the helicity amplitudes can be characterized as the solution to certain extremal problems. The notion of optimal state (Ion, 1982a,b) for each channel $[\mu]$ and the RK of $H^{[\mu]}$ are the same. With this respect the optimal state from the Hilbert space of the helicity amplitudes is the analogous to the coherent state from the Hilbert space of the wave functions.

(3°) For the RKHS $H^{[\mu]}$ the FOS system (36) is a complete orthogonal set. The expansion (37) of the helicity amplitude in terms of the FOS system is an important alternative to the partial wave analysis. The helicity amplitude (37) in this case is an element of $H^{[\mu]}$ of least norm when $f^{[\mu]}(y_i)$ are given in the all points $y_i \in [-1, +1]$, $i = 1, \ldots, N$, where y_i are the zeros of Jacobi polynomials corresponding to $d^{J_\mu+1}_{\mu\nu}(x)$.

(4°) The predictions (31) and (32a,b) are satisfied experimentally to a surprising accuracy (Ion, 1982a) for all pp, $\bar{p}p$, $\pi^{\pm}p$, $K^{\pm}p$ scattering at all energies higher than 2 GeV.

ACKNOWLEDGMENTS

It is a pleasure to thank Professors Abdus Salam and L. Bertocchi as well as the International Atomic Energy Agency for hospitality extended at the International Centre for Theoretical Physics, Trieste, where this paper was started.

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